

# On a Poisson reduction for Gel'fand–Zakharevich manifolds\*

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## Abstract

We formulate and discuss a reduction theorem for Poisson pencils associated with a class of integrable systems, defined on bi-Hamiltonian manifolds, recently studied by Gel'fand and Zakharevich. The reduction procedure is suggested by the bi-Hamiltonian approach to the Separation of Variables problem.

**Key words:** Bi-Hamiltonian manifolds, Hamiltonian reduction, Poisson manifolds.

## 1 Introduction

The aim of this paper is to present and prove some results about Poisson reduction for bi-Hamiltonian manifolds. The methods presented in the paper are an outgrowth of a geometric theory of Separation of Variables, based on the notion of bi-Hamiltonian geometry, introduced in recent years (see, e.g., [16, 2, 5]), which is thoroughly discussed in [7] and [8]. The cornerstones of such a theory are the (related) concepts of  $\omega N$  manifold and of bi-Lagrangian foliation. An  $\omega N$  manifold  $M$  can be viewed as a special bi-Hamiltonian manifold, where one of the two compatible Poisson brackets defined on  $M$  is actually *symplectic*, i.e., is associated with a symplectic 2-form  $\omega$ . A

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bi-Lagrangian foliation  $\mathcal{F}$  defined on an  $\omega N$  manifold  $M$  is a foliation of  $M$  which is *Lagrangian* with respect to both brackets. The content of the “bi-Hamiltonian theorem of Separation of Variables” can be (roughly) summarized as follows:

A Hamiltonian  $H$  is *separable* in a set of coordinates (called Darboux-Nijenhuis coordinates) naturally associated with the  $\omega N$  structure of  $M$  if and only if the Hamiltonian vector field  $X_H$ , defined by the equation

$$\omega(X_H, \cdot) = -dH ,$$

is tangent to a bi-Lagrangian foliation.

Although the notions of Hamiltonian vector field (and, *a fortiori*, of separability) pertain to the category of *symplectic manifolds*, it “experimentally” turns out that a major source of separable systems admit a more natural formulation in a wider class of manifolds, that is, Poisson manifolds. This feature is particularly evident in the case of integrable systems connected with soliton equations and loop algebras (see, e.g., [10, 4, 1, 18, 19]), related to the theory of Lax equations and the  $R$ -matrix formalism.

We will be interested in the class of systems nowadays known as *Gel'fand-Zakharevich* (GZ) systems. These are bi-Hamiltonian systems defined over a bi-Hamiltonian manifold  $M$  where none of the Poisson brackets is symplectic, over which, so to speak, the geometry of the Poisson pencil  $P' - \lambda P$  itself selects a complete family of mutually commuting Hamiltonians. Indeed, by definition, for a (torsionless) GZ system of rank  $k$ , this family can be grouped in  $k$  *Casimirs of the Poisson pencil*. A Casimir of the pencil is a polynomial  $H(\lambda) = H_0\lambda^n + H_1\lambda^{n-1} + \dots + H_n$ , whose coefficients are functions on  $M$ , that satisfy the equation

$$(P' - \lambda P)dH(\lambda) = 0 . \tag{1.1}$$

It is well-known (and quite easy to check) that such equation entails the Lenard recursion relations for the vector fields  $X_i = PdH_{i+1}$ , and, as a consequence, the mutual commutativity with respect to both brackets of the coefficients  $H_i$ . So, the family of the vector fields associated with the  $k$  above-mentioned Casimirs defines a distribution  $\mathcal{X}$  in  $TM$  (called the axis of the pencil) which has the following property: it is tangent to the symplectic leaves  $\bar{S}$  of any fixed element  $\bar{P}$  of the Poisson pencil defined on  $M$ , and, when viewed as a distribution defined on the symplectic manifold  $\bar{S}$  (endowed with the natural symplectic structure induced by  $\bar{P}$ ) it is a Lagrangian distribution. Once we have fixed such a “preferred” element  $\bar{P}$ , we can thus discuss whether we can induce on  $\bar{S}$  another Poisson bracket, starting from another element of the Poisson pencil  $P_\lambda$ . In doing this, we require (for the reasons briefly addressed above, and related to the Separation of Variables problem) to preserve two properties:

- a) compatibility of the reduced brackets on  $\bar{S}$ ;
- b) commutativity of the (restriction to  $\bar{S}$  of the) Hamiltonians  $H_j$ .

To this end, we will have to make an assumption on the pencil, that is, we will suppose that  $P_\lambda$  admits a distribution  $\mathcal{Z}$  transversal to the symplectic leaves of  $\bar{P}$ , enjoying “good” properties (to be discussed later on in the paper).

The problem can be tackled in two ways: from the point of view of Poisson tensors (on a tubular neighbourhood of  $\bar{S}$ ) and from the point of view of induced 2-forms (on the single leaf  $\bar{S}$ ). Since, in our opinion, both ways have their own strong points we decided to divide the paper in two sections, and present both points of view. In particular, the “Poisson” point of view allows for simpler proofs and a nice description in terms of Poisson-Lichnerowicz geometry. The other point of view has the advantage that it clearly points out that the problem “lives” on single symplectic leaves, and frames its study within the scheme of symplectic geometry.

The results contained in this paper have been used (more specifically those concerning what we call the *strong form of the reduction theorem* in Remark 1) in [5, 6, 3].

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## 2 GZ Poisson pencils and their reduction to symplectic leaves

We are interested in a special class of GZ systems, known as complete torsionless rank  $k$  systems of pure Kronecker type. They are studied in, e.g., [12, 11, 17, 20, 22]).

They can be defined as the datum of:

- i) a bi-Hamiltonian manifold  $(M, \{\cdot, \cdot\}_\lambda)$ , that is, a manifold  $M$  endowed with a linear pencil of Poisson brackets  $\{\cdot, \cdot\}_\lambda = \{\cdot, \cdot\}' - \lambda\{\cdot, \cdot\}$  or equivalently with a linear pencil  $P_\lambda = P' - \lambda P$  of Poisson tensors, where

$$\{f, g\}_\lambda = \langle df, P_\lambda dg \rangle.$$

- ii) a collection of  $k$  polynomial Casimir functions  $H^{(1)}, \dots, H^{(k)}$ , that is, a collection of degree  $n_j$  polynomials

$$H^{(j)}(\lambda) = \sum_{i=0}^{n_j} H_i^{(j)} \lambda^{n_j-i}$$

such that:

- a)  $n_1 + n_2 + \dots + n_k = n$ , with  $\dim M = 2n + k$ .
- b) The differentials  $\{dH_s^{(j)}\}_{j=1, \dots, k; s=0, \dots, n_j}$  are linearly independent at every point and so define an  $(n + k)$ -dimensional distribution in  $T^*M$ .

The collection of the  $n$  bi-Hamiltonian vector fields

$$X_k^{(j)} = P d H_{k+1}^{(j)} = P' d H_k^{(j)} \quad (2.1)$$

associated with the Lenard sequences defined by the polynomials Casimir  $H^{(j)}$  is called the *GZ system* associated with the given GZ manifold, or *axis* of the bi-Hamiltonian manifold  $M$ . We now consider (clearly, this specific choice is inessential) the preferred Poisson tensor  $\bar{P}$ , mentioned in the Introduction, to be exactly  $P$ . Let  $S$  be one of its symplectic leaves. We seek for a deformation of the Poisson structure  $P'$  to  $Q = P' + \Delta P'$ , such that the following three properties hold:

1.  $Q$  restricts to  $S$ ;
2.  $Q - \lambda P$  is still a Poisson pencil;
3.  $P' - \lambda P$  and  $Q - \lambda P$  share the same axis.

To this end we consider a distribution  $\mathcal{Z}$  transversal to  $S$  and such that it splits the tangent space to  $M$  as

$$TM = TS \oplus \mathcal{Z} . \quad (2.2)$$

We assume, at this stage, that  $\mathcal{Z}$  is defined in a whole tubular neighbourhood  $U_S$  of  $S$ . We consider the family of Casimir functions  $(H_0^{(1)}, \dots, H_0^{(k)})$  of  $\{\cdot, \cdot\}$  defining  $S$ , and a family of vector fields  $(Z_1, \dots, Z_k)$  spanning  $\mathcal{Z}$ . We can assume that the transversal vector fields  $Z_a$  are normalized:

$$Z_a(H_0^{(b)}) = \delta_a^b . \quad (2.3)$$

We consider the “first” vector fields of the each Lenard sequence, that is,

$$X'_a = P' d H_0^{(a)} ,$$

and we define the new bivector

$$Q = P' - \sum_{a=1}^k X'_a \wedge Z_a . \quad (2.4)$$

**Theorem 2.1** 1) *The bivector  $Q$  restricts to  $S$ , and all the GZ Hamiltonians  $H_l^{(a)}$  are skew orthogonal with respect to the second “bracket” defined by  $Q$ , that is,*

$$\{H_l^{(a)}, H_k^{(b)}\}_Q := \langle dH_l^{(a)}, Q dH_k^{(b)} \rangle = 0 . \quad (2.5)$$

2) *The second bracket satisfies the Jacobi identity if and only if*

$$\sum_{a=1}^k X'_a \wedge \left( L_{Z_a}(P') - \sum_{b=1}^k [Z_a, X'_b] \wedge Z_b \right) - \frac{1}{2} \sum_{a,b=1}^k X'_a \wedge X'_b \wedge [Z_a, Z_b] = 0, \quad (2.6)$$

that is, if and only if

$$\sum_{a=1}^k X'_a \wedge \left( L_{Z_a}(Q) + \frac{1}{2} \sum_{b=1}^k X'_b \wedge [Z_a, Z_b] \right) = 0. \quad (2.7)$$

3) In this case, the two brackets define a Poisson pencil if and only if

$$\sum_{a=1}^k X'_a \wedge L_{Z_a}(P) = 0. \quad (2.8)$$

**Proof.** To prove that  $Q$  restricts to  $S$  we remark that, if we consider bivectors as maps from  $T^*M$  to  $TM$ , the corresponding expression for the map associated with  $Q$  is given by

$$Q(\alpha) = P'(\alpha) + \sum_{a=1}^k (< \alpha, X'_a > Z_a - < \alpha, Z_a > X'_a) .$$

We must show that  $\text{Im}(Q) \subset TS$ , i.e., that for every 1-form  $\alpha$  and  $b = 1, \dots, k$ ,

$$< dH_0^{(b)}, Q(\alpha) > = 0 .$$

This is true thanks to the antisymmetry of  $Q$ , and the fact that  $QdH_0^{(b)} = 0$ . The validity of Eq. (2.5) is proved exactly in the same way, taking into account the commutativity of the vector fields entering the Lenard sequences.

The proof of the last two assertions is done via a computation which makes use of the formal properties of the Schouten brackets of multivectors, and, especially, of the fact that the Schouten bracket is an extension to polyvector fields of the Lie derivative for vector fields (see, e.g. [21]). In particular, we will use the following facts:

- i) If  $X$  and  $Y$  are vector fields, the Schouten bracket  $[X, Y]_S$  coincides with the commutator  $[X, Y]$ .
- ii) the Lie derivative along a vector field  $Z$  of the wedge product of two vector fields  $X$  and  $Y$  satisfies:

$$L_Z(X \wedge Y) = [Z, X] \wedge Y + X \wedge [Z, Y] .$$

- iii) if  $X$  is a vector field and  $P$  a bivector, then

$$[X, P]_S = L_X(P) .$$

- iv) If  $X, Y$  are vector fields and  $P$  is a bivector one has:

$$[X \wedge Y, P]_S = Y \wedge [X, P]_S - X \wedge [Y, P]_S = Y \wedge L_X(P) - X \wedge L_Y(P) .$$

Using i) through iv) one can argue as follows: It is well-known that  $\{\cdot, \cdot\}_Q$  is a Poisson bracket if and only if the Schouten bracket  $[Q, Q]_S$  vanishes, and that the compatibility between  $\{\cdot, \cdot\}_Q$  and  $\{\cdot, \cdot\}$  takes the form  $[Q, P]_S = 0$ . Let us compute

$$\begin{aligned} [Q, Q]_S &= \left[ P' - \sum_{a=1}^k X'_a \wedge Z_a, P' - \sum_{a=1}^k X'_a \wedge Z_a \right]_S \\ &= -2 \sum_{a=1}^k [X'_a \wedge Z_a, P']_S + \sum_{a,b=1}^k [X'_a \wedge Z_a, X'_b \wedge Z_b]_S \\ &= -2 \sum_{a=1}^k (L_{X'_a} P' \wedge Z_a - X'_a \wedge L_{Z_a} P') \\ &\quad + \sum_{a,b=1}^k (2X'_a \wedge [Z_a, X'_b] \wedge Z_b - X'_a \wedge X'_b \wedge [Z_a, Z_b] + [X'_a, X'_b] \wedge Z_a \wedge Z_b) . \end{aligned}$$

But  $L_{X'_a} P' = 0$  and  $[X'_a, X'_b] = 0$ , so that

$$[Q, Q]_S = 2 \sum_{a=1}^k X'_a \wedge \left( L_{Z_a} P' - \sum_{b=1}^k [Z_a, X'_b] \wedge Z_b \right) - \sum_{a,b=1}^k X'_a \wedge X'_b \wedge [Z_a, Z_b] .$$

Therefore  $[Q, Q]_S = 0$  if and only if (2.6) (as well as its equivalent form (2.7)) holds.

As far as assertion 3) is concerned, we have

$$\begin{aligned} [Q, P]_S &= \left[ P' - \sum_{a=1}^k X'_a \wedge Z_a, P \right]_S = - \sum_{a=1}^k [X'_a \wedge Z_a, P]_S \\ &= \sum_{a=1}^k (-L_{X'_a} P \wedge Z_a + X'_a \wedge L_{Z_a} P) . \end{aligned}$$

Now we recall that the compatibility condition between  $P$  and  $P'$  can be written as

$$L_{P'dF} P + L_{PdF} P' = 0 \quad \text{for all } F \in C^\infty(M).$$

Since  $H_0^{(a)}$  is a Casimir of  $P$ , this implies that  $L_{X'_a} P = 0$ . Hence we have

$$[Q, P]_S = \sum_{a=1}^k X'_a \wedge L_{Z_a} P ,$$

and the theorem is proved. ■

**Remark 1** When the transversal distribution  $\mathcal{Z}$  is integrable, Equation (2.6) simplifies to

$$\sum_{a=1}^k X'_a \wedge \left( L_{Z_a}(P') - \sum_{b=1}^k [Z_a, X'_b] \wedge Z_b \right) = 0 . \quad (2.9)$$

So, the conditions for  $Q - \lambda P$  to be a Poisson pencil reduce to

$$\begin{aligned} \sum_{a=1}^k X'_a \wedge L_{Z_a}(Q) &= 0 \\ \sum_{a=1}^k X'_a \wedge L_{Z_a}(P) &= 0 . \end{aligned} \quad (2.10)$$

The “strong” solutions to this system, that is, the distributions  $\mathcal{Z}$  spanned by vector field satisfying, for  $a = 1, \dots, k$ ,

$$\begin{aligned} L_{Z_a} P' &= \sum_{b=1}^k [Z_a, X'_b] \wedge Z_b \\ L_{Z_a} P &= 0 , \end{aligned} \quad (2.11)$$

can be described in the framework of the Marsden–Ratiu reduction scheme [15] for Poisson manifolds. Indeed, one can notice that the conditions (2.11) on the Poisson pencil imply (actually, are equivalent to, see [21]) the fact that the ring of functions which are left invariant by the distribution  $\mathcal{Z}$  are a *Poisson* subalgebra with respect to the pencil. This means that if  $Z(F) = Z(G) = 0$  for every  $Z \in \mathcal{Z}$ , then  $Z(\{F, G\}_\lambda) = 0$ . Thus the bi-Hamiltonian structure can be projected onto every symplectic leaf  $S' \subset U_S$  of  $P$ , and the restriction of  $Q$  to  $S'$  coincides with the *reduction* of  $P'$  to  $S'$ . The reader should however be aware of the fact that, in the general case, that is, when  $Q$  satisfies Equations (2.7), the reduction scheme herewith presented does not fit in the MR setting.

**Remark 2** It is interesting to compare our approach to the reduction of  $P'$  with the classical Dirac reduction procedure for second class constraints. The latter (see, e.g., [15, 21]) is usually described as follows. Let  $(M, \{\cdot, \cdot\})$  be a Poisson (or even symplectic) manifold, and let  $\{\phi_1, \dots, \phi_{2k}\}$  be a family of “constraints” for a Hamiltonian system defined on  $M$ . One says that the constraints are second class if the matrix of Poisson brackets

$$C_{ab} = \{\phi_a, \phi_b\} \quad (2.12)$$

is nondegenerate on a submanifold  $S \subset M$ , where  $S$  is defined by the  $2n$  equations  $\phi_a = \text{const}_a, a = 1, \dots, 2k$ . The Dirac bracket  $\{\cdot, \cdot\}^D$  is defined on  $S$  as follows:

$$\{F, G\}^D = \{F, G\} - \sum_{a,b=1}^n \{F, \phi_a\} (C^{-1})_{ab} \{\phi_b, G\}. \quad (2.13)$$

In terms of Poisson tensors it is not difficult to check that, if  $P$  is associated with  $\{\cdot, \cdot\}$  and  $X_a$  is the Hamiltonian vector field associated with  $\phi_a$ , that is,  $X_a = Pd\phi_a$ , the Poisson tensor associated with  $\{\cdot, \cdot\}^D$  is

$$P^D := P - \frac{1}{2} \sum_{a,b=1}^n X_b \wedge (C^{-1})_{ab} X_a. \quad (2.14)$$

This can be interpreted (see also [21]) as follows. The Dirac bracket is a deformation of the “original one” to one for which the constraint functions (that define the special submanifold  $S$ , or, better, a local foliation given by the constraints  $\{\phi_a\}$ ) are Casimirs. In particular, the analogy with Eq. (2.4) is enhanced by remarking that the vector fields

$$Y_a := \sum_{b=1}^k (C^{-1})_{ab} X_a$$

of Eq. (2.14) are normalized with respect to the Casimirs of  $P^D$ . The fundamental difference between the two instances resides in the fact that, in our case, the Poisson

brackets of the functions  $C_a$  is maximally degenerated, that is, it is the *zero* matrix. So, the choice of the transversal distribution  $\mathcal{Z}$  is a “free input” in our problem. As a consequence of such a freedom, however, we are no longer guaranteed that the reduced “brackets” are Poisson brackets, and so one has to impose on  $\mathcal{Z}$  the condition (2.6).

**Remark 3** The deformation  $P' \rightarrow Q$  defined by Eq. (2.4) is not the unique satisfying the requirements that  $Q$  restricts to  $S$  and that the GZ Hamiltonians are in involution. For instance, one could consider

$$Q' = Q + \Delta$$

with  $\Delta$  a section of the second exterior product of the axis  $\mathcal{X}$  of the GZ manifold. Correspondingly, the requirements that  $Q' - \lambda P$  be a Poisson pencil would take a more complicated form. The choice we made can be considered as a “minimal” one.

**Remark 4** One can notice the following: if Eqs. (2.6) and (2.8) do not hold in the whole tubular neighbourhood  $U_S$  but, say, on a single symplectic leaf  $\bar{S}$ , we can still say that such a single leaf is endowed with a bi-Hamiltonian structure, of regular type, since, by definition,  $P|_{\bar{S}}$  is clearly symplectic. Actually, as we shall see in the next section, one can also drop the assumption that the distribution  $\mathcal{Z}$  be defined in  $U_S$  and require it to exist on a single symplectic leaf. To do that, it is convenient to tackle the problem from a different point of view.

### 3 The $\omega N$ point of view

We want now to discuss the problem at hand from the point of view of symplectic geometry, or, to be more precise, from the point of view of the geometry of  $\omega N$  manifolds. As we have anticipated in Remark 4 above, the advantage of this point of view is that it makes clear that the assumptions about the existence of the transversal distribution in a whole tubular neighbourhood of the symplectic leaf  $S$  are not necessary, and that the reduction process can be discussed, so to say, “leaf by leaf”, even if it involves more complicated computations (which we spare to the reader). To proceed, we need to recall in more details some notions. The basic elements of the concept of  $\omega N$  manifold are a symplectic manifold  $(M, \omega)$ , of dimension  $2n$ , and a second closed 2-form  $\omega'$ . To this form we associate the recursion operator  $N$  defined as

$$\omega'(X, Y) = \omega(NX, Y) . \tag{3.1}$$



**Definition 3.1** *Definition* We say that  $M$  is an  $\omega N$  manifold if the Nijenhuis torsion of  $N$  is vanishing, i.e.,

$$T_N(X, Y) := [NX, NY] - N[NX, Y] - N[X, NY] + N^2[X, Y] = 0 \quad (3.2)$$

for all pairs  $(X, Y)$  of vector fields on  $M$ . In this case we say, for short, that  $N$  is the recursion operator of the manifold  $M$ .

An alternative form of the vanishing condition of the torsion of  $N$  is obtained by introducing the third 2-form  $\omega''$  defined by

$$\omega''(X, Y) = \omega(NX, NY) . \quad (3.3)$$

Indeed, it turns out that the torsion of  $N$  vanishes if and only if  $\omega''$  is closed. To check this, it suffices to use the identity

$$\begin{aligned} d\omega''(X, Y, Z) &= d\omega'(NX, Y, Z) + d\omega'(X, NY, Z) - d\omega(NX, NY, Z) \\ &\quad - \omega(T_N(X, Y), Z) , \end{aligned} \quad (3.4)$$

relating the exterior derivatives of the 2-forms  $\omega$ ,  $\omega'$ ,  $\omega''$  and the torsion  $T_N$  of the recursion operator. The most direct link between the theory of  $\omega N$  manifolds and bi-Hamiltonian manifolds is the following: If  $Q - \lambda P$  is a Poisson pencil on a bi-Hamiltonian manifold  $M$ , and if  $P$  is symplectic, with symplectic form  $\omega$ , the  $(1, 1)$  tensor field

$$N = Q \cdot P^{-1}$$

has vanishing Nijenhuis torsion, so that the pair  $(\omega, N)$  endows  $M$  with the structure of an  $\omega N$  manifold. We refer to [13, 14] for fuller details and proofs.

Let us consider a Poisson manifold  $M$ . It is well-known that such a manifold can be seen as a “glueing” of symplectic leaves. Let  $P$  the Poisson bivector corresponding to the Poisson bracket, that is,  $\{F, G\} = \langle dF, PdG \rangle$ , and let us denote with

$$X_F = PdF \quad (3.5)$$

the Hamiltonian vector field associated with the function  $F$ . As it is well known, the leaves of the distribution generated by the vector fields  $X_F, F \in C^\infty(M)$ , are endowed with the a canonical symplectic form  $\omega$ , explicitly defined as

$$\omega(X_F, X_G) := \{F, G\} . \quad (3.6)$$

Let us suppose now that  $M$  be endowed with a second Poisson bracket, denoted with  $\{\cdot, \cdot\}'$ , which we assume to be compatible with the Poisson bracket associated with  $P$ . In the  $\omega N$  setting, it is thus natural to discuss how to use the second Poisson bracket to induce a second closed 2-form  $\omega'$  on a fixed symplectic leaf  $S$  of the first bracket, in such a way to provide  $S$  with an  $\omega N$  manifold structure. Since the symplectic leaves

of  $\{\cdot, \cdot\}'$  are not contained, in general, in the ones of the first bracket, we cannot use the analog of (3.6) in order to define  $\omega'$ .

To attain our aim, we must be able to “project” in a suitable way the Hamiltonian vector fields  $X'_F$  (that is, the Hamiltonian vector fields associated with  $\{\cdot, \cdot\}'$ ) on the symplectic leaf  $S$ . As the projection process has a local character, without loss of generality we can restrict to an open subset where the functions  $(C_1, \dots, C_k)$  form a basis in the algebra of Casimir functions of  $\{\cdot, \cdot\}$ . In the intersection of such an open set with  $S$  we consider a family of  $k$  vector fields  $Z_a$ ,  $a = 1, \dots, k$ , which, at the points  $p \in S$ , span a subspace of  $T_p M$  complementary (and hence transversal) to  $T_p S$ . We still agree to normalize these vector fields as

$$Z_a(C_b) = \delta_{ab} \quad (3.7)$$

on  $S$ . As in Section 2, we use the vector fields  $Z_a$  to split, in every point  $p$  of  $S$ , the tangent space to  $M$  as the direct sum

$$T_p M = T_p S \oplus \mathcal{Z}_p \quad (3.8)$$

of the tangent space to the leaf and of the distribution spanned by the vector fields  $Z_a$ . Dually, we split the cotangent space as the direct sum

$$T_p^* M = \mathcal{Z}_p^0 \oplus (T_p S)^0 \quad (3.9)$$

of the annihilators. We identify the annihilator  $\mathcal{Z}^0$  of  $\mathcal{Z}$  with the cotangent space of the symplectic leaf, and we denote with  $\pi^*$  the canonical projection on  $\mathcal{Z}^0$  in the splitting (3.9). One has that

$$\pi^*(dF) = dF - \sum_{a=1}^k Z_a(F) dC_a . \quad (3.10)$$

We still denote with  $P'$  the bivector associated with the second Poisson bracket, and we introduce a second 2-form  $\omega'$  on  $S$  according to the relation

$$\omega'(X_F, X_G) := \langle P' \circ \pi^*(dF), \pi^*(dG) \rangle . \quad (3.11)$$

The problem we have to discuss now is how to choose the vector fields  $Z_a$  in such a way that the 2-form  $\omega'$  define an  $\omega N$  structure on the selected leaf  $S$ . We still work under the assumption that Casimirs  $C_a$  of the first bracket form an Abelian algebra with respect to the second bracket, i.e., that  $\{C_a, C_b\}' = 0$ . It implies that the vector fields  $X'_a = P'dC_a$  are tangent to  $S$  and, therefore, it is not necessary to project these vector fields onto  $S$  (whence the simplification of some of the formulae in the sequel). In particular, for  $\omega'$  we obtain the expression

$$\omega'(X_F, X_G) = \{F, G\}' + \sum_{a=1}^k (X'_a(F)Z_a(G) - X'_a(G)Z_a(F)) , \quad (3.12)$$

to be compared with formula (2.4).

According to the definitions given at the beginning of this section, we have to insure that the 2-form  $\omega'$  is closed.

**Theorem 3.2** *The 2-form  $\omega'$  is closed if and only if*

$$\sum_{a=1}^k X'_a \wedge L_{\tilde{Z}_a}(P) = 0 \quad (3.13)$$

at the points of  $S$ , where  $\tilde{Z}_a$  is any extension of the vector field  $Z_a$  (which are in principle defined only on  $S$ ) to a tubular neighbourhood of  $S$  in  $M$ .

**Proof.** We have to compute the exterior derivative of  $\omega'$  on  $S$ . Thanks to the Palais formulae, one has

$$d\omega'(X_F, X_G, X_H) = \sum_{\text{cyclic}} (X_F \omega'(X_G, X_H) - \omega'([X_F, X_G], X_H)) . \quad (3.14)$$

Before proceeding to the computation of this exterior derivative, we recall that the compatibility condition between the two Poisson brackets can be written as

$$\sum_{\text{cyclic}} (\{F, \{G, H\}\}' + \{F, \{G, H\}'\}) = 0 \quad (3.15)$$

for every triple of functions  $(F, G, H)$  on  $M$ . A particular case of this identity is

$$\{F, \{G, C_a\}'\} + \{G, \{C_a, F\}'\} + \{C_a, \{F, G\}'\} = 0 , \quad (3.16)$$

obtained by putting  $H = C_a$ . Now let us substitute the expression of  $\omega'$  in Equation (3.14). Let us use the definition (3.5) of the vector field  $X_F$  and the commutation property  $[X_F, X_G] = X_{\{F, G\}}$  of the Hamiltonian vector fields. Collecting in a suitable way the different terms, using the identities (3.15) and (3.16), we get straight away the identity

$$d\omega'(X_F, X_G, X_H) = \sum_{a=1}^k (X'_a \wedge L_{\tilde{Z}_a}(P)) (dF, dG, dH) . \quad (3.17)$$

It shows that the exterior derivative of the 2-form  $\omega'$ , evaluated on the Hamiltonian vector fields  $X_F, X_G, X_H$ , coincides with the 3-vector  $\sum_{a=1}^k X'_a \wedge L_{\tilde{Z}_a}(P)$  evaluated on the differentials of the corresponding Hamiltonians. This completes the proof. ■

The second step is to check under which additional hypotheses the torsion of the recursion operator  $N$ , associated with  $\omega'$ , vanishes too. To this aim, we will write explicitly  $N$ . From (3.12), taking into account that  $\{C_a, C_b\}' = 0$ , one finds that

$$NX_F = X'_F + \sum_{a=1}^k (X'_a(F)Z_a - Z_a(F)X'_a) . \quad (3.18)$$

**Theorem 3.3** *Suppose that  $d\omega' = 0$  on  $S$ . Then the torsion of  $N$  vanishes if and only if, at the points of  $S$ ,*

$$\sum_{a=1}^k X'_a \wedge R_a = 0, \quad (3.19)$$

where

$$R_a = L_{\tilde{Z}_a}(P' - \sum_{b=1}^k X'_b \wedge \tilde{Z}_b) + \frac{1}{2} \sum_{b=1}^k X'_b \wedge [\tilde{Z}_a, \tilde{Z}_b], \quad (3.20)$$

and, as above,  $\tilde{Z}_a$  is any extension of  $Z_a$  to a tubular neighbourhood of  $S$  in  $M$ .

**Proof.** The proof of this assertion is, in this formalism, rather long, even if in principle not difficult. The reason is that the expression of the torsion of  $N$  contains several terms that must be assembled and managed carefully. Thus, we will give the essential lines of the computation, omitting the details. We begin with computing the expression

$$[X_F, X_G]_N := [NX_F, X_G] + [X_F, NX_G] - N[X_F, X_G]. \quad (3.21)$$

Then we plug in it the form (3.18) of  $NX_F$ , we compute the Lie brackets and we make use once again of (3.15) and (3.16) to simplify the resulting expression. Using also the condition (3.13) of closedness of the 2-form  $\omega'$ , which we already assumed, we arrive in this way at the expression

$$\begin{aligned} [X_F, X_G]_N : &= X_{\{F,G\}'} + \sum_{a=1}^k (Z_a(G)X'_{X'_a(F)} - Z_a(F)X'_{X'_a(G)}) \\ &+ \sum_{a=1}^k (X'_a(F)X'_{Z_a(G)} - X'_a(G)X'_{Z_a(F)}) . \end{aligned} \quad (3.22)$$

It shows that  $[X_F, X_G]_N$  is a linear combination of Hamiltonian (with respect to  $\{\cdot, \cdot\}$ ) vector fields. This make simpler the calculation of  $N[X_F, X_G]_N$ . Finally, we evaluate the Lie bracket  $[NX_F, NX_G]$ . Subtracting  $N[X_F, X_G]_N$  to this Lie bracket, assembling with care the large number of terms of the previous expressions, using the identity

$$[X'_F, X'_G] = X'_{\{F,G\}'} \quad (3.23)$$

and the involutivity of the Casimir functions  $C_a$ , we obtain a still quite complicated expression for the torsion

$$T_N(X_F, X_G) = [NX_F, NX_G] - N[X_F, X_G]_N \quad (3.24)$$

we want to compute. Finally, evaluating the symplectic scalar product

$$\omega(X_H, T_N(X_F, X_G)) = \langle T_N(X_F, X_G), dH \rangle, \quad (3.25)$$

we get the final identity

$$\omega(X_H, T_N(X_F, X_G)) = \sum_{a=1}^k (X'_a \wedge R_a) (dF, dG, dH), \quad (3.26)$$

where  $R_a$  is the expression (3.20). The 3-form in the left-hand side, thanks to (3.4), is the exterior derivative  $d\omega''$  of the 2-form  $\omega''$ . Hence we have the identity

$$d\omega''(X_F, X_G, X_H) = \sum_{a=1}^k (X'_a \wedge R_a) (dF, dG, dH) , \quad (3.27)$$

which proves the theorem. ■

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